From Fractions to Algebra

Preparing for Algebra by Building Fraction Sense

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It is the second week of sixth grade, and Sammy slouches in his desk chair, feeling unsure of his abilities to answer fraction problems on his worksheet. He remembers learning about fractions in previous grades, but he still does not feel confident with simple fractions. He puts pencil to paper and makes his best effort to answer the problems. "Shade in $\frac{1}{3}$ of six boxes." Sammy focuses only on the numerator and shades one of the boxes instead of two. "Here is a 0-to-1 number line. Where does $\frac{1}{2}$ go?" Sammy accurately places $\frac{1}{2}$ in the middle of the 0-to-1 number line. Like most sixth graders, he is very familiar with $\frac{1}{2}$ as one out of two parts, or "the middle." "Here is a 0-to-2 number line. Where does $\frac{1}{2}$ go?" Sammy does not think of $\frac{1}{2}$ as a magnitude on the number line, so he mistakenly places $\frac{1}{2}$ in the middle of the 0-to-2 number line.

Sammy's responses are typical for sixth graders who may have received several years of fraction instruction but still have a weak foundation for understanding fractions. Fraction skills are important for everyday activities, from measuring cups of sugar for a recipe to managing finances. In addition, there are strong links between students' fraction understanding and success with algebra (Booth & Newton, 2012). Even students' overall IQ, family income, and family education do not predict future algebra achievement as strongly as does fraction knowledge (Siegler et al., 2012). Scholars have proposed that a deep understanding of rational numbers is a crucial foundation for learning algebra (Booth, Newton, & Twiss-Garrity, 2014; Wu, 2001). Thus, bolstering students' fraction understanding is a critical step in preparing students for algebraic thinking.

Understanding fractions as magnitudes that can be represented on a number line provides an underlying structure for learning a range of fraction concepts and skills (Siegler, Thompson, & Schneider, 2011). A student who accurately places the fraction $\frac{5}{6}$ on a number line that extends from 0 to 2 understands how the numerator and the denominator work together to create a single magnitude. Booth and colleagues (2014) found that students' fraction magnitude knowledge predicts their improvement in algebra; that is, students who have a stronger understanding of fraction magnitudes when they begin an algebra course learn more of the algebra content than peers who start with weaker fraction magnitude understanding (Booth et al., 2014).

A strong foundation of rational number sense prepares students for more advanced algebraic thinking. Beyond the frequent presence of fractions in algebraic equations (e.g., $\frac{1}{2}x = 24$), fraction knowledge and algebraic thinking share important conceptual underpinnings. For example, a Common Core State Standard in mathematics for operations and algebraic thinking (3OA.B.5) states that students should understand properties of multiplication and the relationship between multiplication and division (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010). When working with fractions, students must see the relationship between multiplication and division as they interpret $\frac{4}{2}$ as "four groups of $\frac{1}{2}$," "two sets of $\frac{2}{2}$," or "$4 \div 2$." This sort of fraction knowledge helps students make sense of solutions to more advanced problems, such as $6 = 12y$ (e.g., "12 times ? = 6" or "$\frac{6}{12} = y$").

Math activities that center on the number line build fraction concepts (e.g., Siegler et al., 2011). As early as third grade, a number line approach for teaching fractions is supported in the Common Core State Standards (NGA & CCSSO, 2010), and this approach has proved helpful for advancing low-performing students' fraction understanding (Fuchs et al., 2013). In this article, we provide examples of how to integrate the number line into fraction activities to address common fraction misunderstandings and to prepare students for later algebra courses.

Barriers to the Development of Fraction Knowledge

A longitudinal study on the development of fraction knowledge revealed three related barriers to fraction learning: (a) focusing on the numerator as a counting number and ignoring the denominator and the related whole, (b) not grasping how the numerator and denominator work together to determine the magnitude of a fraction, and (c) failing to understand that fractions are magnitudes that can be represented on a number line (Jordan, Resnick, Rodrigues, Hansen, & Dyson, 2016).

The example in Figure 1 shows Sammy's tendency to attend only to the numerator when reasoning through a fraction problem. To find $\frac{1}{3}$ of six partitions, he focuses on the numerator and shades only one partition, rather than separating the whole figure into

Figure 1. Misunderstanding the Relationship Between the Numerator and Denominator

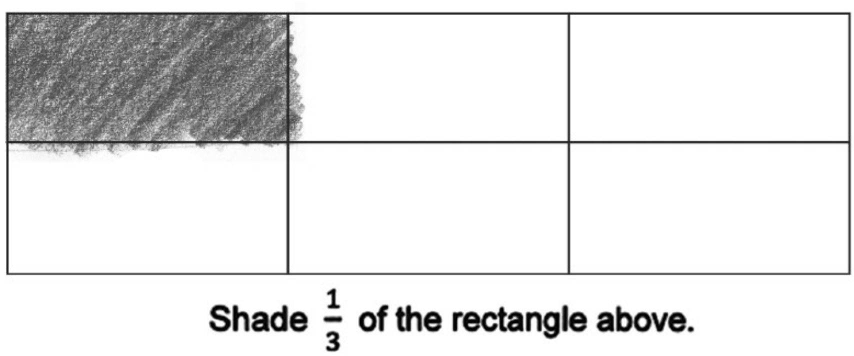
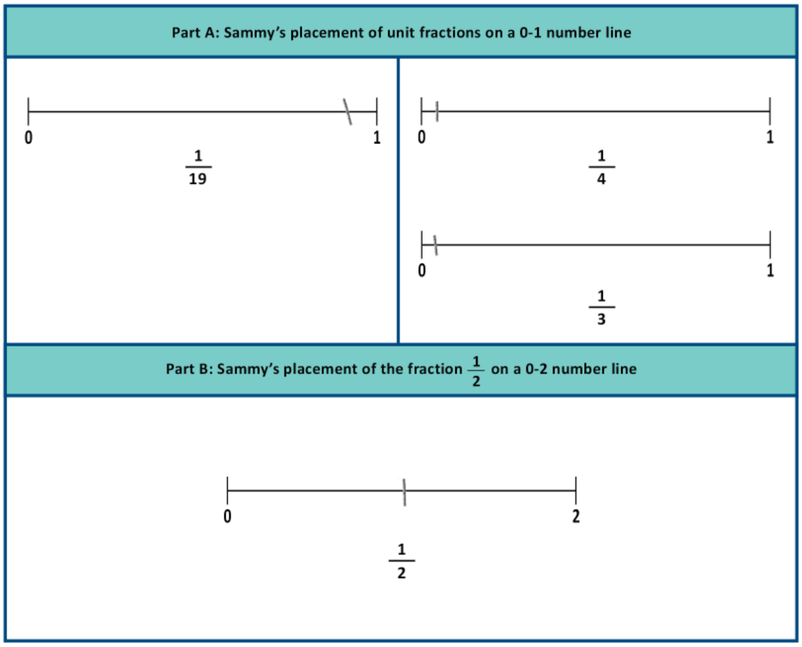


Figure 2. Inaccurate Estimations of Fraction Magnitudes on a Number Line



three equal parts and then shading one of those three (which would be two of the six partitions). His response suggests that he is incorrectly applying his knowledge of the numerator to this fraction problem (e.g., "I see the number 1 in the numerator, so I'll count only one box and shade it in"). Sammy does not grasp the relationship between the numerator and denominator.

Sammy demonstrates his misunderstanding of how the numerator and denominator of a fraction work together to determine a single magnitude when he estimates fraction magnitudes on a number line (Figure 2, Part A). When asked to place the unit fraction $\frac{1}{19}$ on a 0-to-1 number line, he focuses on the "large" denominator 19 and reasons that this fraction must be placed at the far right of the line, close to 1. He then places unit fractions with "smaller" denominators (e.g., $\frac{1}{4}$ and $\frac{1}{3}$) to the far left of the line, in nearly the same exact location on each line.

A different example (Figure 2, Part B) shows Sammy's incorrect placement of the fraction $\frac{1}{2}$ in the middle of a 0-to-2 number line. He locates $\frac{1}{2}$ of the line, rather than $\frac{1}{2}$ of the number 1. Sammy demonstrates a basic understanding of the concept of one half, but he does not attend to the appropriate whole.

More generally, Sammy struggles to think of fractions as numbers that can be placed on a number line. When we asked him to describe what a fraction is, he chimed in immediately, "Fractions have a lot to do with pie . . . edible pie!" Sammy's tendency to think of fractions as pies (e.g., a pizza pie partitioned into pieces) most likely reflects his previous instruction emphasizing part–whole representations of fractions.

Developing Fraction Magnitude Understanding Using the Number Line

These barriers to fraction understanding can be overcome for most struggling students by introducing number line-based activities that focus on the relationship of the numerator to the denominator and how this relationship determines the magnitude of the fraction. Using number lines to support fraction magnitude understanding is an approach supported by the Common Core (NGA & CCSSO, 2010) and prior research (Siegler et al., 2011). However, using number lines to teach fractions has not been a strong focus in many U.S. mathematics classrooms in the past. Many curricula instead emphasize a part–whole representation of fractions (Ni & Zhou, 2005; Thompson & Saldanha, 2003).

Support Understanding Through Meaningful Contexts

Although number lines help students see fractions as numbers with magnitude, this representation may be too abstract for students who have math difficulties. One way to support number line understanding is to anchor it in a meaningful context. Introducing number line activities in real-world contexts provides an engaging environment for learning about fractions as magnitudes. For example, using a race story: Students are sponsoring a race for charity and various stations are set up at fractions of a mile along the race course (e.g., snack stations every half mile, drink stations every fourth mile, flags to mark eighths of a mile, etc.). The length of the race courses can vary but should extend beyond 1 mile (e.g., a 4-mile race course) to support students' understanding of improper fractions and mixed numbers. Once students are comfortable with the number line in the context of a story, students can be presented with a context-free number line.

Table 1. Sample Sequence of Activities for a 45-Minute Fraction Sense Lesson

Activity	Duration (minutes)
Oral counting of fraction magnitudes	3
Number line race course activities *Explicit instruction in fraction magnitude concepts using number line activities*	20
Whole number multiplication fluency practice using flash cards	3
Independent practice *Students complete practice handouts that reflect the fraction concepts taught during explicit instruction*	10
Fraction games using flash cards	5
Formative assessment *Students complete a brief formative assessment that reflects the concepts taught during the day's lesson*	4

Support Understanding Using a Limited Number of Denominators

Teaching core fraction concepts with a limited number of denominators helps students focus on concepts without having to consider several ways of partitioning. Number sense research has shown that struggling kindergartners develop number concepts more easily when concepts are introduced with just a few familiar numbers, such as 1 and 2 (Dyson, Jordan, Beliakoff, & Hassinger-Da, 2015). The same approach works with introducing fraction concepts. The foundational concept of one half (one of two equal parts) is accessible to most students, even very low-achieving ones. Students can explore many fraction concepts using just halves (e.g., $\frac{1}{2}, \frac{2}{2}, \frac{3}{2} \ldots$). Teaching with only the denominator of 2 helps students develop deep knowledge without being overwhelmed with many denominators. Other denominators can be introduced in a logical progression (described in the next section) as students master fraction concepts with each new denominator. Using a limited number of denominators also helps prepare students for everyday activities that use the familiar denominators of 2, 3, 4, and 8, such as measuring with a ruler or a measuring cup.

Activities That Promote Fraction Magnitude Understanding

The following activities have been found to be engaging and helpful for understanding fraction magnitude. A typical lesson sequence of activities is shown in Table 1.

Creating a Number Line Race Course

Students can be introduced to the whole—1 mile—using a paper bar. Using different but related and familiar visual representations solidifies conceptual understanding and encourages flexible thinking (Dienes, 1971). The paper bar "mile" creates race course number lines of various sizes (e.g., 1 mile, 2 miles, 3 miles, etc.) by using the bar to measure, mark, and label whole miles along a straight line. This procedure underscores the key insight that there is an equal distance between each whole number. Students can then fold the paper bar into two equal parts and use this bar to measure half miles along the race course. They work from left to right, drawing a mark and placing a small sticker every half mile (Figure 3). Struggling students often skip whole numbers when counting halves, so adding the stickers helps them count every half. Students then label each half mile with the appropriate fraction.

Creating the race course number line helps students see that fractions have increasing magnitudes as they move to the right and decreasing magnitudes as they move to the left. They also see that there are fractions greater than 1 and that fractions can increase in magnitude indefinitely, just as whole numbers do.

After students finish labeling all the whole miles and half miles on their race course number lines (Figure 3), they can then be directed to notice where the same distance has more than one number label (i.e., at each whole mile). Using these points of equivalence, the multiplicative relationship of halves to wholes is made explicit. "Because there are two halves in a whole, 1 mile equals one group of two half miles or 1×2 half miles $=$ 2 half miles. Likewise, two groups of two half miles or 2×2 half miles $=$ 4 half miles," and so on. To promote fluency in this type of activity, it is helpful to reserve time in the lesson to review multiplication facts.

When students have mastered halves (e.g., partitioning and labeling on a number line; modeling halves using linear, area, and set models; adding and subtracting halves; finding the number of halves in a whole or mixed number), teachers can introduce fourths. Fourths are modeled by taking a paper bar, folding it in half, and separating each half into two equal parts by folding in the ends. Students are directed to demonstrate how this strategy can be applied to finding fourths on the number line. They first mark halves on the number line, label them with small stickers, then find fourths by separating each half mile into two equal-sized parts and place another small sticker at each fourth

Figure 3. A "Race Course" Number Line.

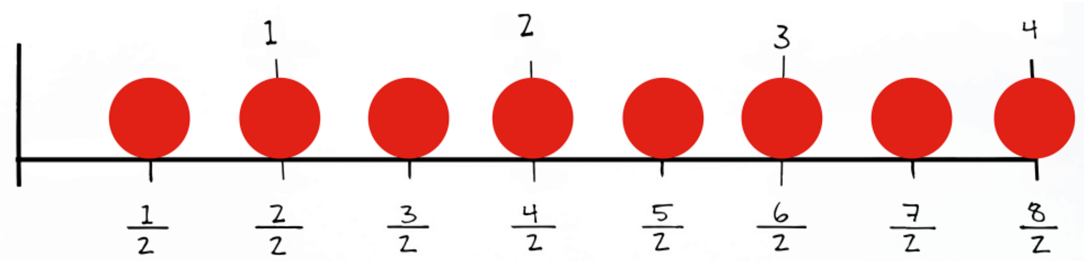
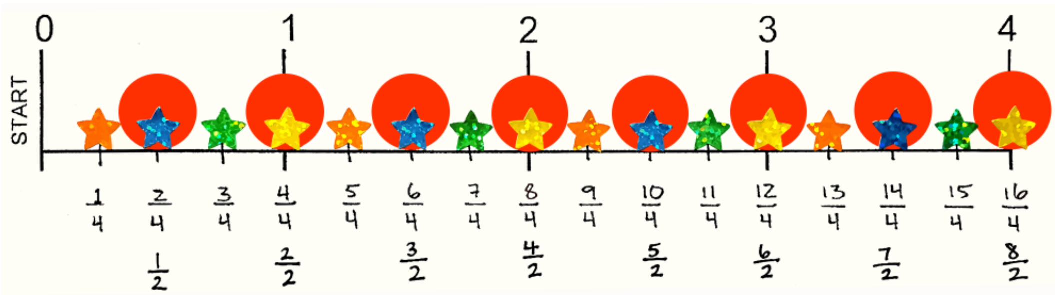


Figure 4. Marking and Labeling Miles Separated into Fourths

Figure 5. Finding Equivalent Fractions on the Race Course Number Line



mark (Figure 4). Students can label each fourth mile with the appropriate fraction and notice where the same distance can be named with a whole number. Again, the multiplicative relationship of fourths to wholes is made explicit (i.e., 4 times the number of whole miles yields the number of fourths of a mile).

In succeeding lessons, students mark and label a race course with both half stickers and fourth stickers to explore equivalence (Figure 5). Each location with two stickers has two number names. The same distance can be named with a number of fourths and a number of halves. Because the students have created fourths by separating each half into two equal parts, it makes sense that it takes twice as many fourths of a mile to cover a distance as it does using halves of a mile. Students can then solve problems finding equivalent halves and fourths using the race course number line they created as a model. This includes adding and subtracting halves and fourths by finding equivalent fractions with like denominators.

Once students become reasonably fluent with halves and fourths, eighths are introduced in the same way (i.e., students find halves, then find fourths, then separate each fourth into two equal parts to create eighths) and equivalent relationships made explicit in a similar manner. Gradual introduction of denominators, with each being created using the previous fractions, means that each lesson is a review of content learned in previous lessons and provides many opportunities to practice past material.

Counting Fraction Magnitudes

Oral counting of increasing fraction magnitudes on a number line supports students' fraction thinking. During this activity, a race course (i.e., number line) labeled with the appropriate fractions is displayed on a whiteboard easel to provide scaffolding for counting (Figure 6). In a group, students count aloud from the smallest magnitude to the largest magnitude using both proper and improper fractions, for example, for fourths, "One fourth, two fourths, three fourths, four fourths, five fourths," and so on. By counting up the 0-to-2 number line, students see that when the numerator is less than the denominator, the fraction is less than 1; when the numerator is equal to the denominator, the fraction equals 1; and when the numerator is greater than the denominator, the fraction is greater than 1. Students can also count using whole and mixed numbers: "One fourth, two fourths, three fourths, one, one and one fourth, one and two fourths, one and three fourths, two." Oral counting allows students to "hear" the multiplicative rhythm of the fraction parts and whole numbers (e.g., every fourth number spoken is a whole number because there are four fourths in a whole).

In later lessons, when teachers introduce eighths, students realize that counting by eighths from 0 to 2 miles takes longer than counting by halves or by fourths. This realization provides a great opportunity to help students think about the meaning of a denominator (e.g., a "larger" denominator creates *more* but *smaller* parts). For example, during Sammy's teacher's oral counting activity, Sammy and his classmate, Mia, spontaneously created their own way of explaining why a "larger" denominator indicates smaller sized partitions.

Moving along the number line race course gives students a concrete way of thinking about fraction addition and subtraction.

Teacher: We counted from 0 to 2 miles by halves, then by fourths, then eighths. Why did it take so much longer to count by eighths?

Sammy: Because of how small they are. You could pretend it's a family. Eighths have so many family members. So, you could pretend that the eighths family is about to eat dinner and cut it into, like, as many pieces of meatloaf so that they have enough for everyone in the family to eat.

Mia: So the meatloaf is cut into small pieces.

Teacher: Yes! If we're talking about the same size meatloaf, and at one dinner, there's only two people splitting it into equal parts and at the other dinner there's eight people splitting it into equal parts, which dinner would you want to be at? Where would you get the bigger piece of meatloaf?

Sammy: The halves! The dinner with two people!

Adding and Subtracting on the Number Line

The number line is also a helpful tool for teaching addition and subtraction of fractions. Using the race context, fraction addition can be modeled as moving to the right on the race course number line and subtraction as moving to the left. Each student is given a magnetic race course in a metal pan and a small magnet that represents a runner participating in the race; they move the magnetic runner to the right for addition or to the left for subtraction (Figure 7). For example, to solve $\frac{3}{4} + \frac{3}{4}$, students place their runner magnet at the $\frac{3}{4}$-mile mark and then move three fourths to the right, landing on $\frac{6}{4}$.

Students perform addition and subtraction "moves" using the magnet and record their solutions on paper. As they move the magnetic runner along the number line, students see when the operation takes them past 1 mile, requiring the solution to be a mixed number or improper fraction. Often, students have no strategy to think about fraction addition and subtraction but instead rely on a rote procedure. Moving along the number line race course gives students who struggle a concrete way of thinking about fraction addition and subtraction. It helps them understand why the denominator must be the same for both fractions to perform the operation. Magnets are motivating for middle school students and reduce frustration because they "stay in place" on the number line.

Using Games to Promote Fluency in Fraction Thinking

Three to 5 minutes of every lesson should be dedicated to practicing concepts and strategies in a way that promotes fluency. The goal is to have students quickly estimate the magnitude of a fraction (e.g., less than, greater than, or equal to 1) and find equivalencies using multiplicative strategies (e.g., how many fourths are in the whole number 2?). Using blank cards and a marker, teachers can make a variety of card games for students to play as a group or individually. Whole numbers as well as fractions used in the lessons are printed on the cards. These cards can be used in the following ways:

Figure 6. Oral Counting From Left to Right on a Visual Number Line

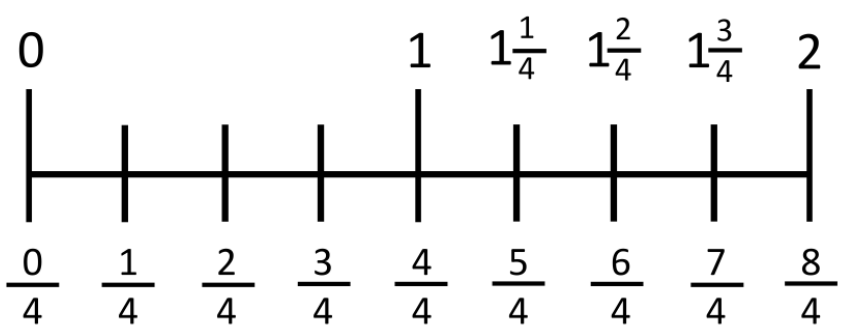


Figure 7. Using Runner Magnets to Solve Fraction Addition and Subtraction Problems

1. Comparing fractions to a benchmark: A fraction comparison game helps students reason quickly about fraction magnitudes and the relations between a fraction's numerator and denominator. Students are shown a card that has a fraction written on it, and they must say whether the fraction is less than, equal to, or greater than 1. For example, students are shown flashcards with both proper and improper fractions that have denominators of 8 (e.g., $\frac{3}{8}$, $\frac{8}{8}$, and $\frac{10}{8}$) as well as denominators of 2 and 4 for review of early lessons. A more challenging version of this game requires students to make magnitude comparisons to $\frac{1}{2}$ rather than 1.
2. Comparing fraction magnitudes: Students begin by comparing magnitudes of two fractions with like denominators and saying aloud the fraction with the greater magnitude. Teachers gradually introduce more challenging activities, such as asking students to compare two or more fractions with unlike denominators using previously taught multiplicative strategies.
3. Whole numbers to fractions: Students are shown a whole-number card (e.g., 3) and must say the number of halves, fourths, or eighths that equal that whole number. The game can be played in reverse where students are shown an improper fraction and must say the whole number it equals. A more advanced version of the game uses mixed numbers.

Next Steps

When students are comfortable working with halves, fourths, and eighths, additional denominators can be introduced. Thirds present an important new challenge because they require students to work with an odd number of parts. The same process of making fourths and eighths from halves can be repeated with thirds, sixths, and 12ths (i.e., students first find thirds, then separate each third into two equal parts to create sixths, then separate each sixth into two equal parts to create 12ths).

Teachers can create other practical fraction magnitude activities that use a limited number of denominators in a meaningful context. For example, students might measure with a ruler that is first marked in halves, then halves and fourths, and finally a typical ruler with halves, fourths, and eighths. Students can also measure with a measuring cup, which can be thought of as a vertical number line. Denominators on measuring cups are commonly limited to halves, fourths, eighths, and thirds.

What's the "Whole" Story?

Fractions are troublesome for many children, especially students with learning difficulties and disabilities in mathematics (Hansen, Jordan, & Rodrigues, in press; Jordan et al., 2016; Mazzocco & Devlin, 2008). To address this serious educational concern, we recommend the use of number lines to build fraction sense. Instructional activities should involve a limited number of denominators and be set in a meaningful context to sustain students' interest. Advancing students' fraction knowledge puts them on a firmer course for algebra success.

References

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, *37*, 247–253. doi:10.1016/j.cedpsych.2012.07.001

Booth, J. L., Newton, K. J., & Twiss-Garrity, L. K. (2014). The impact of fraction magnitude knowledge on algebra performance and learning. *Journal of Experimental Child Psychology*, *118*, 110–118. doi:10.1016/j.jecp.2013.09.001

Dienes, Z. P. (1971). *Building up mathematics*. London, England: Hutchinson Educational.

Dyson, N., Jordan, N. C., Beliakoff, A., & Hassinger-Da, B. (2015). A kindergarten number-sense intervention with contrasting practice conditions for low-achieving children. *Journal for Research in Mathematics Education*, *46*, 331–370. doi:10.5951/jresematheduc.46.3.0331

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., Jordan, N. C., Siegler, R., Gersten, R., & Changas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology*, *105*, 683–700. doi:10.1037/a0032446

Hansen, N., Jordan, N. C., & Rodrigues, J. (in press). Identifying persistent learning difficulties in fractions: A longitudinal study of student growth from third through sixth grade. *Contemporary Educational Psychology*. doi:10.1016/j.cedpsych.2015.11.002

Jordan, N. C., Resnick, I., Rodrigues, J., Hansen, N., & Dyson, N. (2016). The Delaware Longitudinal Study of Fraction Learning: Implications for helping children with mathematics learning difficulties. *Journal of Learning Disabilities*. Advance online publication. doi:10.1177/0022219416662033

Mazzocco, M. M. M., & Devlin, K. T. (2008). Parts and holes: Gaps in rational number sense in children with vs. without mathematical learning disability. *Developmental Science*, *11*, 681–691. doi:10.1111/j.1467-7687.2008.00717.x

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for mathematics*. Washington, DC: Author. Retrieved from http://www.corestandards.org/Math/Content/OA/

Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*, 27–52. doi:10.1207/s15326985ep4001_3

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*, 691–697. doi:10.1177/0956797612440101

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*, 273–296. doi:10.1016/j.cogpsych.2011.03.001

Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. E. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 95–114). Reston, VA: National Council of Teachers of Mathematics.

Wu, H. (2001). How to prepare students for algebra. *American Educator*, *25*(2), 10–17. Retrieved from https://math.berkeley.edu/~wu/wu2001.pdf

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